

Weighted norm estimates for the Semyanistyi fractional integrals and Radon transforms

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Dedicated to Professor Sigurdur Helgason on his 85th birthday

ABSTRACT. Semyanistyi's fractional integrals have come to analysis from integral geometry. They take functions on \mathbb{R}^n to functions on hyperplanes, commute with rotations, and have a nice behavior with respect to dilations. We obtain sharp inequalities for these integrals and the corresponding Radon transforms acting on L^p spaces with a radial power weight. The operator norms are explicitly evaluated. Similar results are obtained for fractional integrals associated to k -plane transforms for any $1 \leq k < n$.

1. Introduction

The Radon transforms play the central role in integral geometry and have numerous applications [3, 7, 14, 15]. In 1960 V.I. Semyanistyi [24] came up with an idea to regard the Radon transform on \mathbb{R}^n as a member of suitable analytic family of fractional integrals $R^\alpha f$, so that for sufficiently good f ,

$$(1.1) \quad R^\alpha f|_{\alpha=0} = Rf.$$

This idea paves the way to a variety of inversion formulas for the Radon transform and has proved to be useful in subsequent developments; see, e.g., [16] and references therein.

We recall basic definitions. Let Π_n be the set of all hyperplanes in \mathbb{R}^n . The Radon transform takes a function $f(x)$ on \mathbb{R}^n to a function

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$(Rf)(\tau) = \int_{\tau} f$ on Π_n . The corresponding Semyanistyi's fractional integrals are defined by

$$(1.2) \quad (R^{\alpha}f)(\tau) = \frac{1}{\gamma_1(\alpha)} \int_{\mathbb{R}^n} f(x) [\text{dist}(x, \tau)]^{\alpha-1} dx,$$

$$(1.3) \quad \gamma_1(\alpha) = 2^{\alpha} \pi^{1/2} \Gamma(\alpha/2) / \Gamma((1-\alpha)/2), \quad \alpha > 0; \alpha \neq 1, 3, \dots,$$

where $\text{dist}(x, \tau)$ is the Euclidean distance between the point x and the hyperplane τ . The normalizing coefficient in (1.2) is inherited from one-dimensional Riesz potentials [13, 20]. More general integrals, when τ is a plane of arbitrary dimension $1 \leq k \leq n-1$, were introduced in [21].

Mapping properties of Radon transforms were studied in numerous publications from different points of view; see, e.g., [1, 2, 4, 5, 6, 9, 12, 15, 17, 18, 25, 28], to mention a few. In Section 2 we suggest an alternative approach which works well in weighted L^p spaces with a radial power weight, yields sharp estimates and explicit formulas for the operator norms, and extends to fractional integrals (1.2). Similar results are obtained in Section 3 for more general operators associated to the k -plane transform for any $1 \leq k \leq n-1$. Main results of the paper are presented by Theorems 2.1, 2.3, 3.2, and 3.3.

Our approach was inspired by a series of works on operators with homogeneous kernels; see, e.g., [8, 23, 27, 29]. A common point for this class of operators and the afore-mentioned operators of integral geometry is a nice behavior with respect to rotations and dilations. The same approach is applicable to the dual Radon transforms and the corresponding dual Semyanistyi integrals; see, e.g., [22], where a different technique has been used.

2. Fractional integrals associated to the Radon transform

2.1. Preliminaries. In the following $\sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere S^{n-1} in \mathbb{R}^n ; $d\sigma(\eta)$ stands for the surface element of S^{n-1} ; e_1, \dots, e_n are coordinate unit vectors; $O(n)$ is the group of orthogonal transformations of \mathbb{R}^n endowed with the invariant probability measure. The L^p spaces, $1 \leq p \leq \infty$, are defined in a usual way, $1/p + 1/p' = 1$. We recall that Π_n denotes the set of all hyperplanes τ in \mathbb{R}^n . Every $\tau \in \Pi_n$ can be parametrized as

$$(2.1) \quad \tau(\theta, t) = \{x \in \mathbb{R}^n : x \cdot \theta = t\}, \quad (\theta, t) \in S^{n-1} \times \mathbb{R}.$$

Clearly,

$$(2.2) \quad \tau(\theta, t) = \tau(-\theta, -t).$$

We set

$$L_\mu^p(\mathbb{R}^n) = \{f : \|f\|_{p,\mu} \equiv \| |x|^\mu f \|_{L^p(\mathbb{R}^n)} < \infty\},$$

$$L_\nu^p(S^{n-1} \times \mathbb{R}) = \{\varphi : \|\varphi\|_{p,\nu}^\sim \equiv \| |t|^\nu \varphi \|_{L^p(S^{n-1} \times \mathbb{R})} < \infty\},$$

where μ and ν are real numbers. Passing to polar coordinates, we have

$$(2.3) \quad \|f\|_{p,\mu} = \left(\sigma_{n-1} \int_0^\infty r^{n-1+\mu p} dr \int_{O(n)} |f(r\gamma e_1)|^p d\gamma \right)^{1/p}, \quad 1 \leq p < \infty,$$

$$(2.4) \quad \|f\|_{\infty,\mu} = \operatorname{ess\,sup}_{r,\gamma} r^\mu |f(r\gamma e_1)|.$$

Similarly,

$$(2.5) \quad \|\varphi\|_{p,\nu}^\sim = \left(\sigma_{n-1} \int_{-\infty}^\infty |t|^{\nu p} dt \int_{O(n)} |\varphi(\gamma e_1, t)|^p d\gamma \right)^{1/p}, \quad 1 \leq p < \infty,$$

$$(2.6) \quad \|\varphi\|_{\infty,\nu}^\sim = \operatorname{ess\,sup}_{t,\gamma} |t|^\nu |\varphi(\gamma e_1, t)|.$$

We write the Radon transform in terms of the parametrization (2.1) as

$$(2.7) \quad (Rf)(\tau) \equiv (Rf)(\theta, t) = \int_{\theta^\perp} f(t\theta + u) d_\theta u,$$

where $\theta^\perp = \{x : x \cdot \theta = 0\}$, $d_\theta u$ denotes the Lebesgue measure on θ^\perp . Similarly,

$$(2.8) \quad (R^\alpha f)(\tau) \equiv (R^\alpha f)(\theta, t) = \frac{1}{\gamma_1(\alpha)} \int_{\mathbb{R}^n} f(x) |t - x \cdot \theta|^{\alpha-1} dx.$$

THEOREM 2.1. *Let $1 \leq p \leq \infty$, $1/p + 1/p' = 1$, $\alpha > 0$. Suppose*

$$(2.9) \quad \nu = \mu - \alpha - (n-1)/p',$$

$$(2.10) \quad \alpha - 1 + n/p' < \mu < n/p'.$$

Then

$$(2.11) \quad \|R^\alpha f\|_{p,\nu}^\sim \leq c_\alpha \|f\|_{p,\mu},$$

where

$$c_\alpha = \|R^\alpha\| = \frac{2^{1/p-\alpha} \pi^{(n-1)/2} \Gamma\left(\frac{n/p' - \mu}{2}\right) \Gamma\left(\frac{1 - \alpha + \mu - n/p'}{2}\right)}{\Gamma\left(\frac{\mu + n/p}{2}\right) \Gamma\left(\frac{\alpha + n/p' - \mu}{2}\right)}.$$

REMARK 2.2. The necessity of (2.9) can be proved using the standard scaling argument; cf. [26, p. 118]. Let $f_\lambda(x) = f(\lambda x)$, $\lambda > 0$. Then

$$\|f_\lambda\|_{p,\mu} = \lambda^{-\mu-n/p} \|f\|_{p,\mu}, \quad \|R^\alpha f_\lambda\|_{p,\nu}^\sim = \lambda^{-\nu-n-\alpha+1/p'} \|R^\alpha f\|_{p,\nu}^\sim.$$

Hence, whenever $\|R^\alpha f\|_{p,\nu}^\sim \leq c \|f\|_{p,\mu}$ with c independent of f , we get $\nu = \mu - \alpha - (n-1)/p'$. As we shall see below, the condition (2.10) is also best possible.

2.2. Proof of Theorem 2.1.

2.2.1. *Step 1.* Let us prove (2.11). For $t > 0$, changing variables $\theta = \gamma e_1$, $\gamma \in O(n)$, and $x = t\gamma y$, we get

$$(2.12) \quad (R^\alpha f)(\gamma e_1, t) = \frac{t^{\alpha+n-1}}{\gamma_1(\alpha)} \int_{\mathbb{R}^n} f(t\gamma y) |1 - y_1|^{\alpha-1} dy.$$

If $1 \leq p < \infty$, then, by Minkowski's inequality, using (2.5) and (2.3), we obtain that the norm $\|R^\alpha f\|_{p,\nu}^\sim$ does not exceed the following:

$$(2.13) \quad \begin{aligned} & \frac{2^{1/p}}{\gamma_1(\alpha)} \int_{\mathbb{R}^n} |1 - y_1|^{\alpha-1} \left(\sigma_{n-1} \int_0^\infty \int_{O(n)} t^{(\alpha+n-1+\nu)p} |f(t\gamma y)|^p d\gamma dt \right)^{1/p} dy \\ & = c_\alpha \|f\|_{p,\mu}, \quad c_\alpha = \frac{2^{1/p}}{\gamma_1(\alpha)} \int_{\mathbb{R}^n} |1 - y_1|^{\alpha-1} |y|^{-\mu-n/p} dy. \end{aligned}$$

If $p = \infty$, then (2.6) and (2.4) give a similar estimate $\|R^\alpha f\|_{\infty,\nu}^\sim \leq c_\alpha \|f\|_{\infty,\mu}$ in which c_α has the same form with $1/p = n/p = 0$.

To evaluate c_α , denoting $\lambda = \mu + n/p$, we have

$$\begin{aligned} c_\alpha &= \frac{2^{1/p}}{\gamma(\alpha)} \int_{-\infty}^\infty |1 - y_1|^{\alpha-1} dy_1 \int_{\mathbb{R}^{n-1}} (|y'|^2 + y_1^2)^{-\lambda/2} dy' = c_1 c_2, \\ c_1 &= 2^{1/p} \int_{\mathbb{R}^{n-1}} (1 + |z|^2)^{-\lambda/2} dz = \frac{2^{1/p} \pi^{(n-1)/2} \Gamma((\lambda + 1 - n)/2)}{\Gamma(\lambda/2)}, \\ c_2 &= \frac{1}{\gamma_1(\alpha)} \int_{-\infty}^\infty |1 - y_1|^{\alpha-1} |y_1|^{n-\lambda-1} dy_1 = \frac{\gamma_1(n - \lambda)}{\gamma_1(\alpha + n - \lambda)}; \end{aligned}$$

see, e.g., [13], where convolutions of Riesz kernels are considered in any dimensions. Combining these formulas with (1.3), we obtain the desired expression. We observe that the integral in (2.13) is finite if and only if $\alpha - 1 + n/p' < \mu < n/p'$, which is (2.10).

2.2.2. *Step 2.* Let us show that $c_\alpha = \|R^\alpha\|$. By Step 1, $\|R^\alpha\| \leq c_\alpha$. Thus, it remains to prove that $\|R^\alpha\| \geq c_\alpha$. Since the operator $R^\alpha : L_\mu^p(\mathbb{R}^n) \rightarrow L_\nu^p(S^{n-1} \times \mathbb{R})$ is bounded, then for any $f \in L_\mu^p(\mathbb{R}^n)$ and $\varphi \in L_{-\nu}^{p'}(S^{n-1} \times \mathbb{R})$ by Hölder's inequality we have

$$(2.14) \quad I = \left| \int_{S^{n-1} \times \mathbb{R}} (R^\alpha f)(\theta, t) \varphi(\theta, t) dt d\theta \right| \leq \|R^\alpha\| \|f\|_{p, \mu} \|\varphi\|_{p', -\nu}^\sim.$$

Suppose $f(x) \equiv f_0(|x|) \geq 0$ and $\varphi(\theta, t) \equiv \varphi_0(|t|) \geq 0$. Then

$$\begin{aligned} I &= \frac{2\sigma_{n-1}}{\gamma_1(\alpha)} \int_0^\infty \varphi_0(t) dt \int_{\mathbb{R}^n} f_0(|x|) |t - x \cdot e_1|^{\alpha-1} dx \\ &= \frac{2\sigma_{n-1}}{\gamma_1(\alpha)} \int_0^\infty \varphi_0(t) dt \int_0^\infty r^{n-1} f_0(r) dr \int_{S^{n-1}} |t - r\eta_1|^{\alpha-1} d\sigma(\eta) \\ &= \frac{2\sigma_{n-1}}{\gamma_1(\alpha)} \int_{S^{n-1}} d\sigma(\eta) \int_0^\infty |1 - s\eta_1|^{\alpha-1} s^{n-1} ds \int_0^\infty \varphi_0(t) f_0(ts) t^{n+\alpha-1} dt. \end{aligned}$$

Let $1 < p < \infty$. Then

$$(2.15) \quad \begin{aligned} \|f\|_{p, \mu} &= \left(\sigma_{n-1} \int_0^\infty r^{n-1+\mu p} f_0^p(r) dr \right)^{1/p}, \\ \|\varphi\|_{p', -\nu}^\sim &= \left(2\sigma_{n-1} \int_0^\infty t^{-\nu p'} \varphi_0^{p'}(t) dt \right)^{1/p'}. \end{aligned}$$

Choose $\varphi_0(t) = t^{\mu p - \alpha} f_0^{p-1}(t)$ so that $\|\varphi\|_{p', -\nu}^\sim = 2^{1/p'} \|f\|_{p, \mu}^{p-1}$. Then (2.14) yields

$$(2.16) \quad \begin{aligned} &\frac{2^{1/p} \sigma_{n-1}}{\gamma_1(\alpha)} \int_{S^{n-1}} d\sigma(\eta) \int_0^\infty |1 - s\eta_1|^{\alpha-1} s^{n-1} ds \\ &\times \int_0^\infty t^{\mu p + n - 1} f_0^{p-1}(t) f_0(ts) dt \leq \|R^\alpha\| \|f\|_{p, \mu}^p. \end{aligned}$$

Finally we set $f_0(t) = 0$ if $t < 1$ and $f_0(t) = t^{-\mu-n/p-\varepsilon}$, $\varepsilon > 0$, if $t > 1$. Then $\|f\|_{p,\mu}^p = \sigma_{n-1}/\varepsilon p$ and (2.16) becomes

$$\begin{aligned} \|R^\alpha\| &\geq \frac{2^{1/p}}{\gamma_1(\alpha)} \int_{S^{n-1}} d\sigma(\eta) \int_0^\infty |1-s\eta_1|^{\alpha-1} s^{n/p'-\mu-1-\varepsilon} \begin{cases} s^{\varepsilon p}, & s < 1 \\ 1, & s > 1 \end{cases} ds \\ &= \frac{2^{1/p}}{\gamma_1(\alpha)} \int_{\mathbb{R}^n} |1-y_1|^{\alpha-1} |y|^{-\mu-n/p-\varepsilon} \begin{cases} |y|^{\varepsilon p}, & |y| < 1 \\ 1, & |y| > 1 \end{cases} dy; \end{aligned}$$

cf. (2.13). Passing to the limit as $\varepsilon \rightarrow 0$, we obtain $\|R^\alpha\| \geq c_\alpha$.

If $p = 1$, then $\nu = \mu - \alpha$. We choose $\varphi_0(t) = t^{\mu-\alpha}$ and proceed as above. If $p = \infty$, we choose $f_0(r) = r^{-\mu}$. Then $\|f\|_{\infty,\mu} = 1$,

$$I = \frac{2\sigma_{n-1}}{\gamma_1(\alpha)} \int_{S^{n-1}} d\sigma(\eta) \int_0^\infty |1-s\eta_1|^{\alpha-1} s^{n-\mu-1} ds \int_0^\infty \varphi_0(t) t^{n+\alpha-\mu-1} dt,$$

and $I \leq \|R^\alpha\| \|\varphi\|_{1,-\nu}^\sim$. We set $\varphi_0(t) = 0$ if $t < 1$ and $\varphi_0(t) = t^{-\delta}$ if $t > 1$, where δ is big enough. This gives

$$\frac{1}{\gamma_1(\alpha)} \int_{\mathbb{R}^n} |1-y_1|^{\alpha-1} |y|^{-\mu} dy \leq \|R^\alpha\|,$$

or $c_\alpha \leq \|R^\alpha\|$, as desired. \square

2.3. The case $\alpha = 0$. This case corresponds to the Radon transform (2.7) and requires independent consideration.

THEOREM 2.3. *Let $1 \leq p \leq \infty$, $1/p + 1/p' = 1$,*

$$(2.17) \quad \nu = \mu - (n-1)/p', \quad \mu > n/p' - 1.$$

Then $\|Rf\|_{p,\nu}^\sim \leq c \|f\|_{p,\mu}$, where

$$(2.18) \quad c = \|R\| = \frac{2^{1/p} \pi^{(n-1)/2} \Gamma\left(\frac{1+\mu-n/p'}{2}\right)}{\Gamma\left(\frac{\mu+n/p}{2}\right)}.$$

PROOF. The necessity of (2.17) can be checked as in Theorem 2.1. To prove the norm inequality, setting $\theta = \gamma e_1$, $\gamma \in O(n)$, $t > 0$, we have

$$(Rf)(\theta, t) = \int_{\theta^\perp} f(t\theta + u) d_\theta u = t^{n-1} \int_{\mathbb{R}^{n-1}} f(t\gamma(e_1 + z)) dz.$$

Hence,

$$\begin{aligned}
||Rf||_{p,\nu}^{\sim} &= \left(2\sigma_{n-1} \int_0^\infty \int_{O(n)} \left| \int_{\mathbb{R}^{n-1}} t^{(n-1+\nu)p} |f(t\gamma(e_1 + z))|^p dz \right| d\gamma dt \right)^{1/p} \\
&\leq (2\sigma_{n-1})^{1/p} \int_{\mathbb{R}^{n-1}} \left(\int_0^\infty \int_{O(n)} |f(t\gamma(e_1 + z))|^p t^{(n-1+\nu)p} d\gamma dt \right)^{1/p} dz \\
&\quad (\text{set } t|e_1 + z| = t(1 + |z|^2)^{1/2} = s) \\
&= c \left(\int_0^\infty \int_{S^{n-1}} |f(s\eta)|^p s^{(n-1+\nu)p} d\sigma(\eta) ds \right)^{1/p} = c ||f||_{p,\mu},
\end{aligned}$$

where

$$c = 2^{1/p} \int_{\mathbb{R}^{n-1}} \frac{dz}{(1 + |z|^2)^{(\nu+n-1/p)/2}} = 2^{1/p} \int_{\mathbb{R}^{n-1}} \frac{dz}{(1 + |z|^2)^{(\mu+n/p)/2}}.$$

The last integral gives an expression in (2.18). The proof of the equality $c = ||R||$ mimics that in Section 2.2.2. \square

3. Fractional integrals associated to the k -plane transforms

We denote by $\Pi_{n,k}$ the set of all nonoriented k -dimensional planes in \mathbb{R}^n , $1 \leq k \leq n-1$. To parameterize such planes and define the corresponding analogues of R and R^α , we introduce the Stiefel manifold $V_{n,n-k} \sim O(n)/O(k)$ of $n \times (n-k)$ real matrices, the columns of which are mutually orthogonal unit n -vectors. For $v \in V_{n,n-k}$, dv stands for the left $O(n)$ -invariant probability measure on $V_{n,n-k}$ which is also right $O(n-k)$ -invariant. Every plane $\tau \in \Pi_{n,k}$ can be parameterized by

$$(3.1) \quad \tau(v, t) = \{x \in \mathbb{R}^n : v^T x = t\}, \quad (v, t) \in V_{n,n-k} \times \mathbb{R}^{n-k},$$

where v^T stands for the transpose of the matrix v . The case $k = n-1$ agrees with (2.1). Clearly,

$$(3.2) \quad \tau(v, t) = \tau(v\omega^T, \omega t) \quad \forall \omega \in O(n-k).$$

This equality is a substitute for (2.2) for all $1 \leq k \leq n-1$.

The k -plane transform takes a function f on \mathbb{R}^n to a function $(R_k f)(\tau) = \int_\tau f$ on $\Pi_{n,k}$. In terms of the parametrization (3.1) it has the form

$$(3.3) \quad (R_k f)(v, t) = \int_{v^\perp} f(vt + u) dv u,$$

where v^\perp denotes the k -dimensional linear subspace orthogonal to v and $d_v u$ stands the usual Lebesgue measure on v^\perp . Similarly,

$$(3.4) \quad (R_k^\alpha f)(v, t) = \frac{1}{\gamma_{n-k}(\alpha)} \int_{\mathbb{R}^n} f(x) |v^T x - t|^{\alpha+k-n} dx,$$

where $|v^T x - t|$ denotes for the Euclidean norm of $v^T x - t$ in \mathbb{R}^{n-k} ,

$$(3.5) \quad \gamma_{n-k}(\alpha) = \frac{2^\alpha \pi^{(n-k)/2} \Gamma(\alpha/2)}{\Gamma((n-k-\alpha)/2)}.$$

Here $1/\gamma_{n-k}(\alpha)$ is the same normalizing coefficient as in the Riesz potential on \mathbb{R}^{n-k} [13, 20].

REMARK 3.1. One can regard $\Pi_{n,k}$ as a fiber bundle over the Grassmann manifold of all k -dimensional linear subspaces of \mathbb{R}^n . The corresponding parametrization of k -planes is different from (3.1); cf. [21, 22]. In the present article we prefer to work with parametrization (3.1) because it reduces to (2.1) when $k = n - 1$.

Let

$$L_\nu^p(V_{n,n-k} \times \mathbb{R}^{n-k}) = \{\varphi : \|\varphi\|_{p,\nu}^\sim \equiv \| |t|^\nu \varphi \|_{L^p(V_{n,n-k} \times \mathbb{R}^{n-k})} < \infty\},$$

where the L^p norm on $V_{n,n-k} \times \mathbb{R}^{n-k}$ is taken with respect to the measure $dv dt$;

$$v_0 = \begin{bmatrix} I_{n-k} \\ 0 \end{bmatrix} \in V_{n,n-k},$$

I_{n-k} is the identity $(n-k) \times (n-k)$ matrix. Then

$$(3.6) \quad \|\varphi\|_{p,\nu}^\sim = \left(\sigma_{n-k-1} \int_0^\infty r^{\nu p + n - k - 1} dr \int_{O(n)} d\gamma \int_{O(n-k)} |\varphi(\gamma v_0, r \omega e_1)|^p d\omega \right)^{1/p},$$

if $1 \leq p < \infty$, and

$$(3.7) \quad \|\varphi\|_{\infty,\nu}^\sim = \operatorname{ess\,sup}_{r,\gamma,\omega} |r|^\nu |\varphi(\gamma v_0, r \omega e_1)|.$$

THEOREM 3.2. Let $1 \leq p \leq \infty$, $1/p + 1/p' = 1$, $\alpha > 0$. Suppose

$$(3.8) \quad \nu = \mu - \alpha - k/p',$$

$$(3.9) \quad \alpha + k - n/p < \mu < n/p'.$$

Then $\|R_k^\alpha f\|_{p,\nu}^\sim \leq c_{\alpha,k} \|f\|_{p,\mu}$, where

$$c_{\alpha,k} = \|R_k^\alpha\| = 2^{-\alpha} \pi^{k/2} \left(\frac{\sigma_{n-k-1}}{\sigma_{n-1}} \right)^{1/p} \frac{\Gamma\left(\frac{n/p' - \mu}{2}\right) \Gamma\left(\frac{\mu + n/p - k - \alpha}{2}\right)}{\Gamma\left(\frac{\mu + n/p}{2}\right) \Gamma\left(\frac{n/p' - \mu + a}{2}\right)}.$$

3.1. Proof of Theorem 3.2.

3.1.1. *Step 1.* The necessity of (3.8) is a consequence of the equalities

$$\|f_\lambda\|_{p,\mu} = \lambda^{-\mu-n/p} \|f\|_{p,\mu}, \quad \|R_k^\alpha f_\lambda\|_{p,\nu}^\sim = \lambda^{-\nu-\alpha-k/p'-n/p} \|R_k^\alpha f\|_{p,\nu}^\sim,$$

where $f_\lambda(x) = f(\lambda x)$, $\lambda > 0$. Let $\gamma \in O(n)$ and $\omega \in O(n-k)$ be such that $v = \gamma v_0$, $t = |t|\omega e_1$. Changing variables in (3.4) by setting

$$x = |t|\gamma\tilde{\omega}y, \quad \tilde{\omega} = \begin{bmatrix} \omega & 0 \\ 0 & I_k \end{bmatrix},$$

we obtain

$$(3.10) \quad (R_k^\alpha f)(\gamma v_0, |t|\omega e_1) = \frac{|t|^{\alpha+k}}{\gamma_{n-k}(\alpha)} \int_{\mathbb{R}^n} f(|t|\gamma\tilde{\omega}y) |v_0^T y - e_1|^{\alpha+k-n} dy.$$

Suppose first $1 \leq p < \infty$. Then, by Minkowski's inequality, owing to (3.6) and (3.8), the norm $\|R_k^\alpha f\|_{p,\nu}^\sim$ does not exceed the following:

$$\begin{aligned} & \left(\sigma_{n-k-1} \int_0^\infty \int_{O(n)} \int_{O(n-k)} r^{\nu p + n - k - 1} |(R_k^\alpha f)(\gamma v_0, r\omega e_1)|^p d\omega d\gamma dr \right)^{1/p} \\ & \leq \frac{\sigma_{n-k-1}^{1/p}}{\gamma_{n-k}(\alpha)} \int_{\mathbb{R}^n} |v_0^T y - e_1|^{\alpha+k-n} \\ & \quad \times \left(\int_0^\infty \int_{O(n)} \int_{O(n-k)} |f(r\gamma\tilde{\omega}y)|^p r^{\mu p + n - 1} d\omega d\gamma dr \right)^{1/p} dy = c_{\alpha,k} \|f\|_{p,\mu}, \\ (3.11) \quad c_{\alpha,k} & = \left(\frac{\sigma_{n-k-1}}{\sigma_{n-1}} \right)^{1/p} \frac{1}{\gamma_{n-k}(\alpha)} \int_{\mathbb{R}^n} |v_0^T y - e_1|^{\alpha+k-n} |y|^{-\mu-n/p} dy. \end{aligned}$$

If $p = \infty$ we similarly have $\|R_k^\alpha f\|_{\infty,\nu}^\sim \leq c_{\alpha,k} \|f\|_{\infty,\mu}$, where $c_{\alpha,k}$ has the same form as above with $1/p = n/p = 0$.

To compute $c_{\alpha,k}$, we set $\lambda = \mu + n/p$. Then

$$c_{\alpha,k} = \frac{\sigma_{n-k-1}^{1/p}}{\sigma_{n-1}^{1/p} \gamma_{n-k}(\alpha)} \int_{\mathbb{R}^{n-k}} |y' - e_1|^{\alpha+k-n} dy' \int_{\mathbb{R}^k} (|y'|^2 + |y''|^2)^{-\lambda/2} dy'' = c_1 c_2,$$

where

$$c_1 = \left(\frac{\sigma_{n-k-1}}{\sigma_{n-1}} \right)^{1/p} \int_{\mathbb{R}^k} (1+|z|^2)^{-\lambda/2} dz = \left(\frac{\sigma_{n-k-1}}{\sigma_{n-1}} \right)^{1/p} \frac{\pi^{k/2} \Gamma((\lambda - k)/2)}{\Gamma(\lambda/2)},$$

$$c_2 = \frac{1}{\gamma_{n-k}(\alpha)} \int_{\mathbb{R}^{n-k}} |y' - e_1|^{\alpha+k-n} |y'|^{k-\lambda} dy' = \frac{\gamma_{n-k}(n - \lambda)}{\gamma_{n-k}(n - \lambda + \alpha)};$$

see also (3.5). It remains to put these formulas together and make obvious simplifications. Note that the repeated integral in the expression for $c_{\alpha,k}$ is finite if and only if $\alpha + k - n/p < \mu < n/p'$, which is (3.9). Thus $\|R_k^\alpha\| \leq c_{\alpha,k}$.

3.1.2. *Step 2.* To prove that $\|R_k^\alpha\| \geq c_{\alpha,k}$ we follow the reasoning from Section 2.2.2. For any $f \in L_\mu^p(\mathbb{R}^n)$ and $\varphi \in L_{-\nu}^{p'}(V_{n,n-k} \times \mathbb{R}^{n-k})$,

$$I = \left| \int_{V_{n,n-k} \times \mathbb{R}^{n-k}} (R_k^\alpha f)(v, t) \varphi(v, t) dt dv \right| \leq \|R_k^\alpha\| \|f\|_{p,\mu} \|\varphi\|_{p',-\nu}.$$

Let $f(x) \equiv f_0(|x|) \geq 0$, $\varphi(v, t) \equiv \varphi_0(|t|) \geq 0$. Then

$$\begin{aligned} I &= \frac{1}{\gamma_{n-k}(\alpha)} \int_{\mathbb{R}^{n-k}} \varphi_0(|t|) |t|^{\alpha+k} dt \int_{\mathbb{R}^n} f_0(|t||y|) |v_0^T y - e_1|^{\alpha+k-n} dy \\ &= \frac{\sigma_{n-k-1}}{\gamma_{n-k}(\alpha)} \int_{S^{n-1}} d\sigma(\eta) \int_0^\infty |sv_0^T \eta - e_1|^{\alpha+k-n} s^{n-1} ds \int_0^\infty \varphi_0(r) f_0(rs) r^{n+\alpha-1} dr. \end{aligned}$$

Suppose $1 < p < \infty$. Then $\|f\|_{p,\mu}$ can be computed by (2.15) and

$$\|\varphi\|_{p',-\nu} = \left(\sigma_{n-k-1} \int_0^\infty r^{-\nu p' + n - k - 1} \varphi_0^{p'}(r) dr \right)^{1/p'}.$$

cf. (3.6). Choose $\varphi_0(r) = r^{\mu p - \alpha} f_0^{p-1}(r)$. Then

$$\|\varphi\|_{p',-\nu} = \left(\frac{\sigma_{n-k-1}}{\sigma_{n-1}} \right)^{1/p'} \|f\|_{p,\mu}^{p-1}$$

and we have

$$\begin{aligned}
 & \frac{\sigma_{n-k-1}}{\gamma_{n-k}(\alpha)} \int_{S^{n-1}} d\sigma(\eta) \int_0^\infty |sv_0^T \eta - e_1|^{\alpha+k-n} s^{n-1} ds \int_0^\infty f_0^{p-1}(r) f_0(rs) r^{\mu p+n-1} dr \\
 (3.12) \quad & \leq \left(\frac{\sigma_{n-k-1}}{\sigma_{n-1}} \right)^{1/p'} \|R_k^\alpha\| \|f\|_{p,\mu}^p.
 \end{aligned}$$

Setting $f_0(r) = 0$ if $r < 1$ and $f_0(r) = r^{-\mu-n/p-\varepsilon}$, $\varepsilon > 0$, if $r > 1$, we obtain $\|f\|_{p,\mu}^p = \sigma_{n-1}/\varepsilon p$, and therefore,

$$\begin{aligned}
 \|R_k^\alpha\| & \geq \left(\frac{\sigma_{n-k-1}}{\sigma_{n-1}} \right)^{1/p} \frac{1}{\gamma_{n-k}(\alpha)} \int_{S^{n-1}} d\sigma(\eta) \\
 & \times \int_0^\infty |sv_0^T \eta - e_1|^{\alpha+k-n} s^{n-1-\mu-n/p-\varepsilon} \left\{ \begin{array}{ll} s^{\varepsilon p}, & s < 1 \\ 1, & s > 1 \end{array} \right\} ds \\
 & = \left(\frac{\sigma_{n-k-1}}{\sigma_{n-1}} \right)^{1/p} \frac{1}{\gamma_{n-k}(\alpha)} \int_{\mathbb{R}^n} \frac{|v_0^T y - e_1|^{\alpha+k-n}}{|y|^{\mu+n/p+\varepsilon}} \left\{ \begin{array}{ll} |y|^{\varepsilon p}, & |y| < 1 \\ 1, & |y| > 1 \end{array} \right\} dy;
 \end{aligned}$$

cf. (3.11) Passing to the limit as $\varepsilon \rightarrow 0$, we obtain $\|R_k^\alpha\| \geq c_{\alpha,k}$. The cases $p = 1$ and $p = \infty$ are treated as in Section 2.2.2.

3.2. Weighted norm estimates for the k -plane transform.

The following statement deals with the k -plane transform (3.3) and formally corresponds to $\alpha = 0$ in Theorem 3.2.

THEOREM 3.3. *Let $1 \leq p \leq \infty$, $1/p + 1/p' = 1$. Suppose that*

$$(3.13) \quad \nu = \mu - k/p', \quad \mu > k - n/p.$$

Then $\|R_k f\|_{p,\nu} \leq c_k \|f\|_{p,\mu}$, where

$$(3.14) \quad c_k = \|R_k\| = \pi^{k/2} \left(\frac{\sigma_{n-k-1}}{\sigma_{n-1}} \right)^{1/p} \frac{\Gamma\left(\frac{\mu+n/p-k}{2}\right)}{\Gamma\left(\frac{\mu+n/p}{2}\right)}.$$

PROOF. As before, the conditions (3.13) are sharp. To prove the norm inequality, as in Section 3.1.1 we set $v = \gamma v_0$, $t = r\omega e_1$, $\gamma \in O(n)$, $\omega \in O(n-k)$, $r > 0$. This gives

$$(R_k f)(\gamma v_0, r\omega e_1) = r^k \int_{\mathbb{R}^k} f(r\gamma\tilde{\omega}(e_1 + z)) dz, \quad \tilde{\omega} = \begin{bmatrix} \omega & 0 \\ 0 & I_k \end{bmatrix}.$$

If $1 \leq p < \infty$, then combining (3.6) with Minkowski's inequality, we majorize $\|R_k f\|_{p,\nu}^\sim$ by the following:

$$\begin{aligned}
& \left(\sigma_{n-k-1} \int_0^\infty \int_{O(n)} \int_{O(n-k)} r^{\nu p + n - k - 1} |(R_k f)(\gamma v_0, r \omega e_1)|^p d\omega d\gamma dr \right)^{1/p} \\
& \leq \sigma_{n-k-1}^{1/p} \int_{\mathbb{R}^k} \left(\int_0^\infty \int_{O(n)} \int_{O(n-k)} |f(r\gamma\tilde{\omega}(e_1 + z))|^p r^{\mu p + n - 1} d\omega d\gamma dr \right)^{1/p} dz \\
& = c_k \|f\|_{p,\mu}, \\
& c_k = \left(\frac{\sigma_{n-k-1}}{\sigma_{n-1}} \right)^{1/p} \int_{\mathbb{R}^k} (1 + |z|^2)^{-(\mu + n/p)/2} dz.
\end{aligned}$$

The last integral was computed in the previous section. In the case $p = \infty$ we similarly have $\|R_k f\|_{\infty,\nu}^\sim \leq c_k \|f\|_{\infty,\mu}$ with $1/p = n/p = 0$. The proof of the equality $c_k = \|R_k\|$ mimics that in Theorem 2.1. \square

References

1. A. P. Calderón, *On the Radon transform and some of its generalizations*. Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981), 673–689, Wadsworth Math. Ser., Wadsworth, Belmont, CA, 1983.
2. M. Christ, *Estimates for the k -plane transform*, Indiana Univ. Math. J. **33** (1984), 891–910.
3. S. R. Deans, *The Radon transform and some of its applications*, Dover Publ. Inc., Mineola, New York, 2007.
4. S. W. Drury, *A survey of k -plane transform estimates*, Contemp. Math. **91**, Amer. Math. Soc., Providence, RI, (1989), 43–55.
5. J. Duoandikoetxea, V. Naibo, and O. Oruetxebarria, *k -plane transforms and related operators on radial functions*, Michigan Math. J. **49** (2001), 265–276.
6. K.J. Falconer, *Continuity properties of k -plane integrals and Besicovitch sets*, Math. Proc. Camb. Philos. Soc. **87**(2), 221–226.
7. S. Helgason, *Integral geometry and Radon transform*, Springer, 2011.
8. N. K. Karapetyants, *Integral operators with homogeneous kernels*. (Russian) Reports of the extended sessions of a seminar of the I. N. Vekua Institute of Applied Mathematics, Vol. I, no. 1 (Russian) (Tbilisi, 1985), 98–101, 246, Tbilis. Gos. Univ., Tbilisi, 1985.
9. A. Kumar and S. K. Ray, *Mixed norm estimate for Radon transform on weighted L^p spaces*, Proc. Indian Acad. Sci. Math. Sci., **120**, (2010), 441–456.
10. ———, *Weighted estimates for the k -plane transform of radial functions on Euclidean spaces*, Israel J. of Math. **188** (2012), 25–56.
11. ———, *End point estimates for Radon transform of radial functions on Non-Euclidean spaces*, arXiv:1205.1193, Preprint, 2012.

12. I. Laba and T. Tao, *An x-ray transform estimate in \mathbb{R}^n* , Rev. Mat. Iberoam. **17**(2), (2001), 375-407.
13. N. S. Landkof, Foundations of modern potential theory. Translated from the Russian by A. P. Doohovskoy. Die Grundlehren der mathematischen Wissenschaften, Band 180. Springer-Verlag, New York-Heidelberg, 1972.
14. A. Markoe, Analytic Tomography, Encyclopedia of Mathematics and its Applications **106**, Cambridge Univ. Press, 2006.
15. F. Natterer, *The mathematics of computerized tomography*, SIAM, Philadelphia, 2001.
16. E. Ournycheva and B. Rubin, *Semyanistyi's integrals and Radon transforms on matrix spaces*, Journal of Fourier Analysis and Applications, **14** (2008), 60-88.
17. D. M. Oberlin and E. M. Stein, *Mapping properties of the Radon transform*, Indiana Univ. Math. J. **31** (1982), 641-650.
18. E. T. Quinto, *Null spaces and ranges for the classical and spherical Radon transforms*, J. Math. Anal. Appl. **90** (1982), 408-420.
19. Radon, J., *Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten*, Ber. Verh. Sächs. Akad. Wiss. Leipzig, Math. - Nat. Kl., **69** (1917), 262-277.
20. B. Rubin, Fractional integrals and potentials, Pitman Monographs and Surveys in Pure and Applied Mathematics, **82**, Longman, Harlow, 1996.
21. ———, *Reconstruction of functions from their integrals over k -dimensional planes*, Israel J. of Math. **141** (2004), 93-117.
22. ———, *Weighted norm inequalities for k -plane transforms*, arXiv:1207.5180v1, (to appear in Proceedings of the AMS).
23. S. G. Samko, *Proof of the Babenko-Stein theorem* (Russian) Izv. Vyssh. Uchebn. Zaved. Matematika (1975), no. 5(156), 47-51.
24. V.I. Semyanistyi, *On some integral transformations in Euclidean space* (Russian), Dokl. Akad. Nauk SSSR, **134** (1960), 536-539.
25. D. C. Solmon, *A note on k -plane integral transforms*, J. of Math. Anal. and Appl. **71** (1979), 351-358.
26. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, NJ, 1970.
27. R. S. Strichartz, *L^p -estimates for integral transforms*, Trans. Amer. Math. Soc. **136** (1969), 33-50.
28. ———, *L^p -estimates for Radon transforms in euclidean and non-euclidean spaces*, Duke Math. J. **48** (1981), 699-727.
29. T. Walsh, *On L_p estimates for integral transforms*, Trans. Amer. Math. Soc. **155** (1971), 195-215.

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